Lecture 1

Fundamental Concepts of Transport Theory

1 Phase Space

To describe the transport of radiation, we must be able to specify both the position and velocity of particles. A point in phase space corresponds to both a position and a velocity. We will use $\overrightarrow{P}$ to denote a vector of phase-space coordinates. Assuming the use of Cartesian spatial coordinates, the standard set of phase-space coordinates for particle transport is

$$\overrightarrow{P} \equiv (x, y, z, \overrightarrow{\Omega}, E) \, , \tag{1}$$

where x, y, and z are the usual Cartesian coordinates of the particle position, $\overrightarrow{\Omega}$ is a unit Cartesian vector representing the direction of particle flow, and E is the particle energy. The direction coordinates are illustrated in Fig. 1. Note from Fig. 1 that $\overrightarrow{\Omega}$ can be represented in terms of a polar angle, θ, and an azimuthal angle, Φ. It can also be represented in terms of any two of its Cartesian components, which are given by

$$\Omega_x = \sin\theta \, \cos\Phi \, , \tag{2a}$$

$$\Omega_y = \sin\theta \, \sin\Phi \, , \tag{2b}$$

$$\Omega_z = \cos\theta \, . \tag{2c}$$

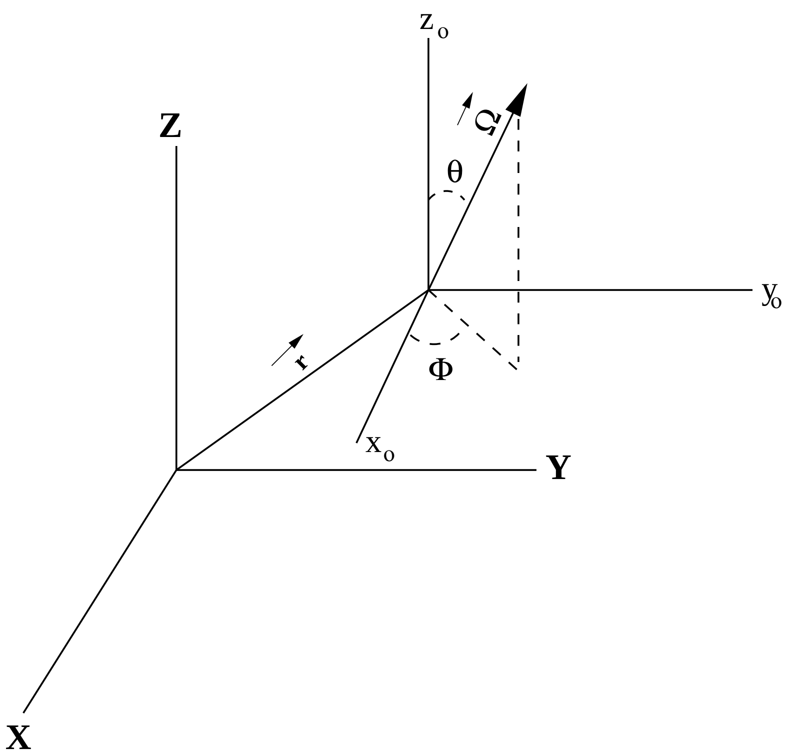

Figure 1: $\overrightarrow{r}$ is the particle position vector, $\overrightarrow{\Omega}$ is the particle direction vector, and E is the particle energy

The differential phase-space volume associated with phase-space point $\overrightarrow{P}$ is

$$dP \equiv dV \, d\Omega dE \,, \tag{3a}$$

where

$$dV = dx \, dy \, dz \,, \tag{3b}$$

and

$$d\Omega = \sin\theta \, d\theta \, d\phi \,. \tag{3c}$$

Note that each point on the unit sphere represents a direction, thus $d\Omega$, represents a differential area on the unit sphere. Since the integral over all directions is equal to 4π, i.e.,

$$\int_0^{2\pi} \int_0^{\pi} \sin\theta \, d\theta \, d\Phi = 4\pi \, , \tag{4}$$

we say that the unit sphere has a total “area” of 4π *steradians*.

A solid angle, which is illustrated in Fig. 2, is a set of points on the unit sphere centered about some direction, $\overrightarrow{\Omega}_0$, with an associated area in steradians, $\Delta\Omega$. Every direction within the solid angle, $\overrightarrow{\Omega}$, satisfies

$$\overrightarrow{\Omega} \cdot \overrightarrow{\Omega}_0 \leq 1 - \Delta\Omega/2\pi \, . \tag{5}$$

Note that $\overrightarrow{\Omega} \cdot \overrightarrow{\Omega}_0$ represents the cosine of the angle subtended by $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega}_0$. A solid angle of 4π contains all directions, A solid angle of π contains half of all the directions, and a solid angle of zero contains only $\overrightarrow{\Omega}_0$.

2 Fundamental Transport Functions

In this section, we define certain fundamental functions associated with transport theory that relate to particle distribution functions. We make a fluid-like approximation in transport theory in that we assume continuum particle distributions even though real particle distributions are discrete in nature. Nonetheless, this assumption is useful because it enables us to use the language of partial integro-differential equations to formulate an equation for particle transport distributions. Furthermore, the continuum distributions that are obtained accurately predict reality if the time

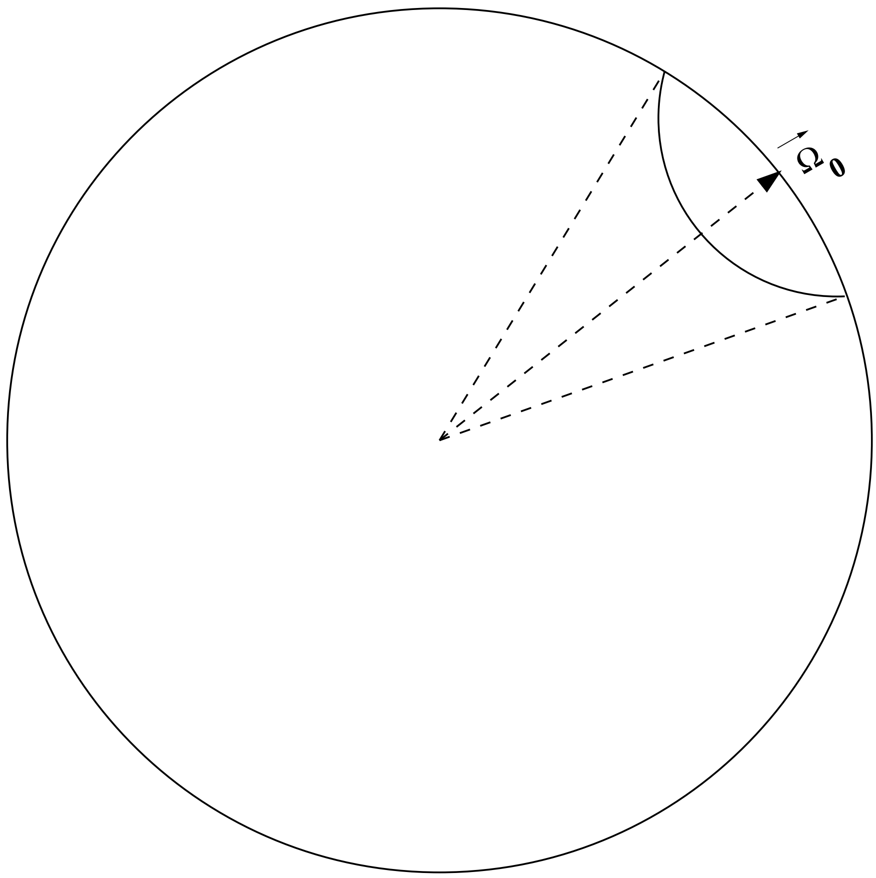

Figure 2: Illustration of solid angle centered about the direction vector, $\vec{\Omega}_0$.

and scales associated with these solutions are large with respect to the time scales associated with particle interactions, and the length scales associated with these solutions are large with respect to particle sizes and interaction lengths. Furthermore, the number of particles in any region in time and space over which a measurement is made must be statistically meaningful to observe agreement with theory.

In the definitions that follow, we often use the word “about” as in “energy about E.” A quantity that is “about” a particular value lies in the differential vicinity of that value. Therefore "a particle with an energy about E" implies a particle that has an energy between E and $E + dE$.

Name: Phase-space Particle Density.

Units: $(particles/(cm^3 - steradian - MeV)$.

Symbol: $\mathcal{N}(\overrightarrow{P})$.

Interpretation: The quantity,

$$\mathcal{N}(\overrightarrow{P})dP\,,$$

represents the number of particles in the differential phase-space volume dP.

Name: Angular Flux.

Symbol: $\psi(\overrightarrow{P})$.

Units: $(particles/(cm^2 - sec - steradian - MeV)$.

Equivalence: $\psi(\overrightarrow{P}) \equiv \mathcal{N}(\overrightarrow{P})v$, where v is the particle speed.

Interpretation 1: Consider a differential surface area, dA, located at position, $\overrightarrow{r}$, and oriented such that it is normal to the direction vector, $\overrightarrow{\Omega}$. Then the expression

$$\psi(\overrightarrow{P})\, dA\, d\Omega\, dE\,,$$

represents the number of particles with directions about $\overrightarrow{\Omega}$ and energies about E, passing per second through dA. The normal orientation of dA is very important.

Interpretation 2: The expression,

$$\psi(\overrightarrow{P})\, dP\,,$$

represents the total pathlength traveled per second by particles within the differential phase-space volume, dP.

Name: Scalar Flux.

Symbol: $\phi(\overrightarrow{r}, E)$.

Units: $(particles/(cm^2 - sec - MeV)$.

Equivalence: $\phi(\overrightarrow{r}, E) \equiv \int_{4\pi} \psi(\overrightarrow{P})\, d\Omega$.

Interpretation: The expression,

$$\phi(\overrightarrow{r}, E)\, dV\, dE\ ,$$

represents the total pathlength traveled per second by particles with energies about E within the differential phase-space volume, dP.

Name: Angular Current Vector.

Symbol: $\overrightarrow{H}(\overrightarrow{P})$.

Units: $(particles/(cm^2 - sec - steradian - MeV)$.

Equivalence 1: $\overrightarrow{H}(\overrightarrow{P}) \equiv \mathcal{N}(\overrightarrow{P})\overrightarrow{v}$, where $\overrightarrow{v}$ is the particle velocity.

Equivalence 2: $\overrightarrow{H}(\overrightarrow{P}) \equiv \psi(\overrightarrow{P})\overrightarrow{\Omega}$.

Interpretation: Consider a differential surface area, dA, located at position, $\overrightarrow{r}$, that is arbitrarily oriented. Every differential surface has two unit normal vectors that are directed in opposition to each other. Let $\overrightarrow{n}_+$ denote the unit vector normal to dA that satisfies $\overrightarrow{\Omega} \cdot \overrightarrow{n} \geq 0$. Then the expression,

$$\overrightarrow{H}(\overrightarrow{P}) \cdot \overrightarrow{n}_+ \, dA \, d\Omega \, dE \, ,$$

represents the number of particles with directions about $\overrightarrow{\Omega}$ and energies about E, passing per second through dA.

Name: Net Current Vector.

Symbol: $\overrightarrow{J}(\overrightarrow{r}, E)$.

Units: $(particles/(cm^2 - sec - MeV)$.

Equivalence: $\overrightarrow{J}(\overrightarrow{r}, E) \equiv \int_{4\pi} \overrightarrow{J}(\overrightarrow{P}) \, d\Omega$.

Interpretation: Consider a differential surface area located at position $\overrightarrow{r}$ that is arbitrarily oriented. Label its two unit normals, $\overrightarrow{n}_L$ and $\overrightarrow{n}_R$, as "left" and "right" as illustrated in Fig. 3. Further let $\Delta\Omega_L$ and $\Delta\Omega_R$ denote those directions that satisfy $\overrightarrow{\Omega} \cdot \overrightarrow{n}_L > 0$, and $\overrightarrow{\Omega} \cdot \overrightarrow{n}_R > 0$, respectively. Then the expression,

$$\int_{\Delta\Omega_L} \overrightarrow{J}(\overrightarrow{P}) \cdot \overrightarrow{n}_L \, dA \, d\Omega \, dE \, ,$$

represents the number of particles with energies about E, passing per second from right-to-left through dA; and the quantity,

$$\int_{\Delta\Omega_R} \overrightarrow{J}(\overrightarrow{P}) \cdot \overrightarrow{n}_R \, dA \, d\Omega \, dE \, ,$$

represents the number of particles with energies about E, passing per second from left-to-right through dA. Since $\overrightarrow{n}_L = -\overrightarrow{n}_R$, it follows that

$$\int_{4\pi} \overrightarrow{J}(\overrightarrow{P}) \cdot \overrightarrow{n}_R \, dA \, d\Omega \, dE = \int_{\Delta\Omega_R} \overrightarrow{J}(\overrightarrow{P}) \cdot \overrightarrow{n}_R \, dA \, d\Omega \, dE - \int_{\Delta\Omega_L} \overrightarrow{J}(\overrightarrow{P}) \cdot \overrightarrow{n}_L \, dA \, d\Omega \, dE \, .$$

Thus we can make the following general statement. Given an arbitrarily oriented differential surface, dA, located at position, $\overrightarrow{r}$, and given a unit vector, $\overrightarrow{n}$, normal to dA, the expression,

$$\int_{4\pi} \overrightarrow{J}(\overrightarrow{P}) \cdot \overrightarrow{n} \, dA \, d\Omega \, dE \, ,$$

represents the "net" particle flow rate through dA due to particles with energies about E, i.e., the flow rate due to particles traveling in directions that satisfy $\overrightarrow{\Omega} \cdot \overrightarrow{n} > 0$ *minus* the flow rate due to particles traveling in directions that satisfy $\overrightarrow{\Omega} \cdot \overrightarrow{n} < 0$.

3 Fundamental Properties of Transport Media

Particles can undergo a wide variety of interactions as they propagate through matter. For simplicity, we consider only absorption and scattering here. However, more types of interactions can be accomodated within the basic framework associated with absorption and scattering. The purpose of this section is to define certain basic functions that describe the transport properties of materials through which the radiation propagates. A statistical viewpoint is taken with respect to

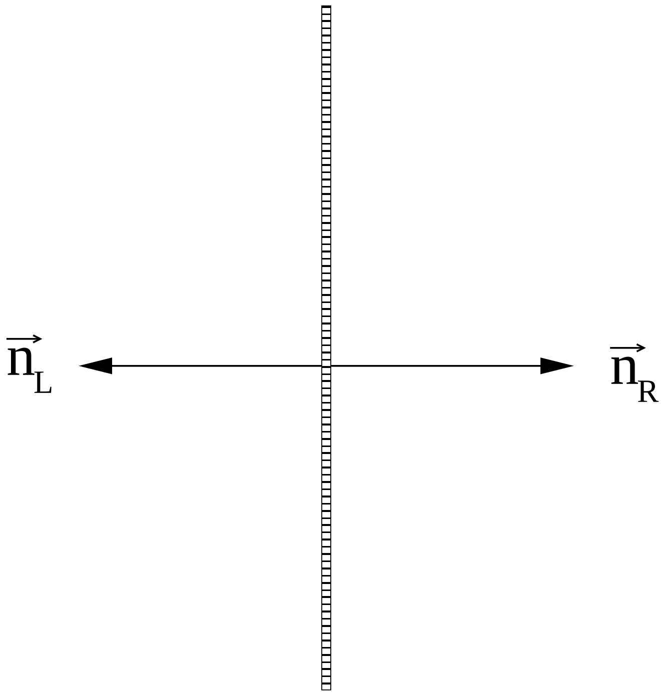

Figure 3: Side-on view of surface with left and right unit normal vectors.

interactions, i.e., over a given pathlength, the number of interactions that a particle will undergo is probabilistic rather than deterministic.

Name: Microscopic Interaction Cross-Section.

Symbol: $\sigma(\overrightarrow{r}, E)$.

Units: cm^2.

Interpretation: This is the effective cross-sectional area of a target atom for a particular type of interaction seen at position, $\overrightarrow{r}$, by a transport particle with an energy about E.

Name: Macroscopic Interaction Cross-Section.

Symbol: $\Sigma(\overrightarrow{r}, E)$.

Units: cm^{-1}.

Equivalence: $\Sigma \overrightarrow{r}, E) = \rho_a(\overrightarrow{r})\sigma(\overrightarrow{r}, E)$, where ρ_a is the atomic density $(atoms/cm^3)$.

Interpretation 1: The expression,

$$\Sigma(\overrightarrow{r}, E)\, ds\,,$$

represents the expected number of interactions by a particle with energy, E, that starts at position, $\overrightarrow{r}$, and travels a differential distance, ds.

Interpretation 2: Assuming a space-independent value of Σ, the quantity,

$$\frac{1}{\Sigma}\,,$$

represents the average distance that a particle will travel between interactions. This distance is called the *mean-free-path*.

Name: Differential Scattering Distribution Function.

Symbol: $f(\overrightarrow{r}, E' \to E, \theta_s)$.

Units: $(steradian^{-1} - MeV^{-1}$.

Interpretation: Consider the scattering frame coordinate system illustrated in Fig. 4. Given that a particle with initial energy, E', has scattered at position, $\overrightarrow{r}$, the expression,

$$f(\overrightarrow{r}, E' \to E, \theta_s)\, d\Omega_s\, dE\,,$$

represents the probability that the particle will scatter into the differential solid angle, $d\Omega_s =$

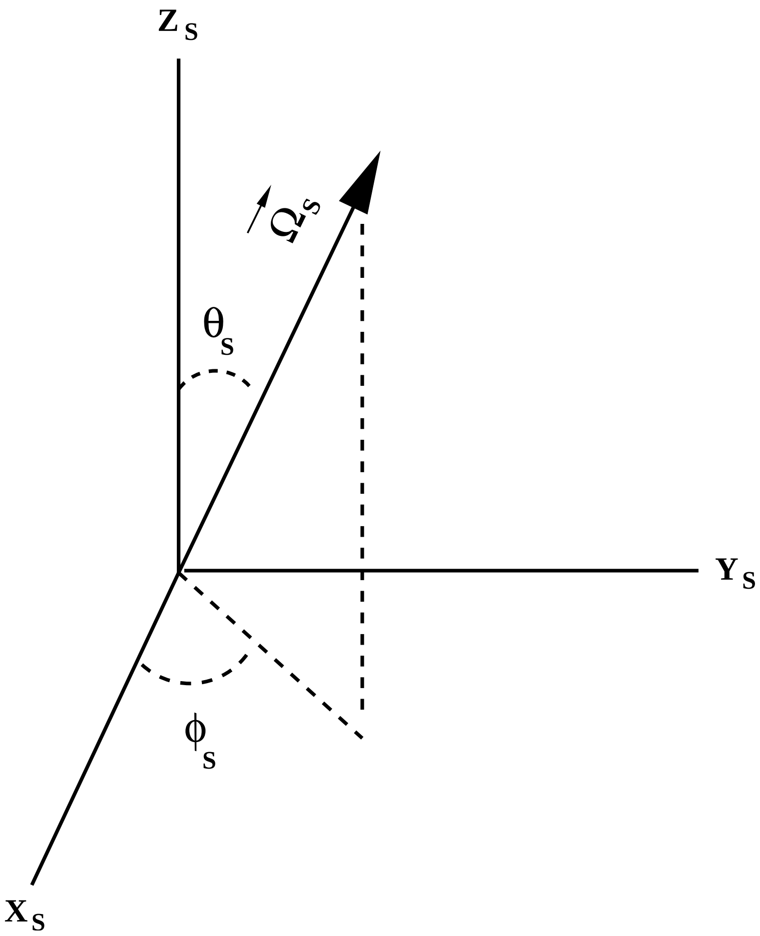

Figure 4: The scattering frame coordinate system. The z_s-axis lies along the initial direction of the scattering particle. The angle, θ_s, is called the polar scattering angle. The angle, ϕ_s, is called the azimuthal scattering angle, and the vector, $\overrightarrow{\Omega}_s$, represents the direction into which the particle scatters.

$\sin\theta_s\, d\theta_s\, d\phi_s$, with a final energy about E. Note that f is independent of the azimuthal scattering angle. Since f is a probability distribution function, it follows that

$$\int_0^\infty 2\pi \int_0^\pi f(\overrightarrow{r}, E' \to E, \theta_s)\, d\theta_s\, dE = 1\,.$$

Name: The Macroscopic Differential Scattering Cross-Section.

Symbol: $\Sigma_s(\overrightarrow{r}, E' \to E, \theta_s)$.

Units: $(cm^{-1} - steradian^{-1} - MeV^{-1})$.

Equivalence: $\Sigma_s(\overrightarrow{r}, E' \to E, \theta_s) = \Sigma_s(\overrightarrow{r}, E) f(\overrightarrow{r}, E' \to E, \theta_s)$.

Interpretation: The expression,

$$\Sigma_s(\overrightarrow{r}, E' \to E, \theta_s)\, ds\, d\Omega_s\, dE\,,$$

represents the expected number of scattering events by a particle with initial energy, E', that starts at position, $\overrightarrow{r}$, travels a differential distance, ds, and scatters into the differential solid angle, $d\Omega_s$, with a final energy about, E.

Name: Differential Scattering Kernel.

Symbol: $\mathcal{K}_s(\overrightarrow{r}, E' \to E, \overrightarrow{\Omega}' \to \overrightarrow{\Omega})$.

Units: $(steradian^{-1} - MeV^{-1}$.

Equivalence: $\mathcal{K}_s(\overrightarrow{r}, E' \to E, \overrightarrow{\Omega}' \to \overrightarrow{\Omega}) = \Sigma_s(\overrightarrow{r}, E' \to E, \overrightarrow{\Omega}' \cdot \overrightarrow{\Omega})$.

Interpretation: The expression,

$$\mathcal{K}_s(\overrightarrow{r}, E' \to E, \overrightarrow{\Omega}' \to \overrightarrow{\Omega})\, ds\, d\Omega\, dE\,,$$

represents the expected number of scattering events by a particle with initial energy, E', and initial direction, $\overrightarrow{\Omega}'$, that starts at position, $\overrightarrow{r}$, travels a differential distance, ds, and scatters into a direction about $\overrightarrow{\Omega}$, with a final energy about, E.

4 Reaction Rates and Sources

We use fundamental transport functions together with fundamental transport medium properties to obtain particle reaction rates and certain solution-dependent sources.

Name: Total Interaction Rate.

Representation: $\psi(\overrightarrow{P})\sigma_t(\overrightarrow{r}, E)$.

Units: $(p/(cm^3 - sec - steradian - MeV)$.

Interpretation: The expression,

$$\psi(\overrightarrow{P})\,\sigma_t(\overrightarrow{r}, E)\;dP\,,$$

represents the number of particles per second being absorbed and scattered within the differential volume dV with directions about $\overrightarrow{\Omega}$ and energies about, E.

Name: Scattering Source.

Symbol: $S(\overrightarrow{P})$.

Representation: $\int_0^\infty \int 4\pi\,\mathcal{K}_s(\overrightarrow{r}, E' \to E, \overrightarrow{\Omega}' \to \overrightarrow{\Omega})\,\psi(\overrightarrow{r}, \overrightarrow{\Omega}', E')\;d\Omega'\,dE'\,.$

Units: $(p/(cm^3 - sec - steradian - MeV)$.

Interpretation: The expression,

$$S(\overrightarrow{P})\,\sigma_t(\overrightarrow{r}, E)\;dP\,,$$

represents the number of particles per second within the differential volume dV being scattered into directions about $\overrightarrow{\Omega}$ and energies about, E.

Name: Inhomogeneous Source.

Symbol: $Q(\overrightarrow{P})$.

Units: $(p/(cm^3 - sec - steradian - MeV)$.

Interpretation: The expression,

$$Q(\overrightarrow{P})\,\sigma_t(\overrightarrow{r}, E)\; dP\,,$$

represents the number of particles being created per second within the differential volume dV with directions about $\overrightarrow{\Omega}$ and energies about, E.